A-STAR SUMMER CAMP ASSESSMENT EXAM
AIME, USAMO/USAJMO LEVELS

INSTRUCTIONS

(1) This is a 25-question, 4-hour examination. However, you may exceed 2 hours if needed.
(2) All answers are integers.
(3) No aids are permitted other than scratch paper, graph paper, rulers, protractors, and erasers. No calculators are allowed. No problems on the test will *require* the use of a calculator.
(4) SCORING: Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
(5) Level Recommendation*:
  - Around 0-4 points: Consider taking AMC10/12 Class
  - Around 5-14 points: AIME Class
  - Around 15-25 points: USAMO/USAJMO Class

  * During the camp, adjustments can be made to the level (depending on space availability) if A-Star Faculty and student agrees that a different level would be more beneficial for the student.

QUESTIONS

(1) In a 21-question exam correct answers are awarded by 4 points, incorrect answers are penalized by 1 point and no point is given for blank answers. At most how many students took the exam if all the students ended up with different total scores?

(2) Find the product $(1+\sqrt{2}+\sqrt{3})(-1+\sqrt{2}+\sqrt{3})(1-\sqrt{2}+\sqrt{3})(1+\sqrt{2}-\sqrt{3})$.

(3) Each surface of a cube is painted via a different color using six colors. Two colorings are considered same if one could be obtained from the other through rotations. How many different colorings are there?

(4) In quadrilateral $ABCD$, $\angle A = \angle D = 90°$. M is the midpoint of the side DC. Given that AC is perpendicular to BM, $DC = 12$ and $AB = 9$, find AD.

(5) How many positive integer divisors of 6^{12} are multiples of 12^6?

(6) One night, more than 1/3 of the students of a school go to the movies, more than 3/10 of them go to theater, and more than 4/11 of them go to a concert. At least how many students are there at the school?

(7) In how many ways can we partition a set of 6 elements into 3 disjoint non-empty subsets?

(8) $ABCD$ is a square. M and N are points on sides BC and CD, respectively. If $BM = 21$, $DN = 4$, and $NC = 24$, find the angle $\angle MAN$ in degrees.

(9) For how many 5-digit numbers is any digits, except the first and the last one, the same as the sum of its two adjacent digits modulo 5?

(10) Find the sum of all the real solutions to the equation

$$(2 + (2 + (2 - x)^2)^2)^2 = 2000.$$

(11) We write all possible 512 sequences of 9-letter sequences using the letters a and p. How many of these contain the word *papa*?

(12) $ABCD$ is a square. E and F are points on line segments BC and ED, respectively so that $DF = BF$ and $EF = BE$. Find the angle $\angle DFA$, in degrees.

(13) For a positive integer n, let $d(n)$ be the largest odd divisor of n. Find the last three digits of the sum

$$d(1) + d(2) + d(3) + \ldots + d(2^{99}).$$

(14) Find the number of ordered triples (x, y, z) of real numbers satisfying the following system of equations:

$$\begin{aligned}(x+y)^5 &= z\\(y+z)^5 &= x\\(z+x)^5 &= y\end{aligned}$$

(15) At most how many subsets can we choose from a set with 10 elements so that no two of these subsets contain one another?

(16) In triangle ABC, $AB = 6$, $BC = 7$, and $CA = 8$. D is a point on side BC so that AD bisects angle A. E is a point on side CA so that $CE = 2$. Find DE.

(17) Let $\omega_1, \omega_2, \omega_3, \ldots, \omega_{21}$ be the solutions of $x^{21} = 1$. Let

$$S = \sum_{i=1}^{200} \sum_{j=1}^{21} \omega_j^i.$$

Evaluate S.

(18) Let N be the number of 10-digit decimal numbers with all different digits that are multiples of $11,111$. Find the last three digits of N.

(19) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. How many functions $f : A \to A$ are there so that $f(f(a)) = a$ for all $a \in A$?

(20) In how many ways can we select 4 different numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ so that no two are consecutive.

(21) In triangle ABC, $\angle C = 90°$. Let D be the center of the inscribing circle of ABC. Let N be the point of intersection of the lines AD and CB. If $CA + AD = CB$ and $CN = 2$, find NB.

(22) Let x and y be two positive integers such that $x^2 + ny^2$ is divisible by 7. For how many positive integers of n less than 100 is it true that x must necessarily be divisible by 7?

(23) How many consecutive nines occur immediately after the decimal point for the number $(3 + \sqrt{11})^{400}$?

(24) Let A be the sum of the reciprocals of the positive integers that can be formed by only using the digits 0,1,2,3. That is

$$A = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{20} + \frac{1}{21} + \frac{1}{22} + \frac{1}{23} + \frac{1}{30} + \dots$$

What is $\lfloor A \rfloor$?

(25) Points $A = (0, 0)$, $B = (3, 0)$, and $C = (0, 4)$ are given. Consider the locus of points X for which $5AX = 3CX + 4BX$. What is the length of this locus to the nearest integer?